2. Real Numbers

- **Commutativity**
  - Whole numbers are commutative under addition and multiplication. However, they are not commutative under subtraction and division.
  - Integers are commutative under addition and multiplication. However, they are not commutative under subtraction and division.
  - Rational numbers:
    1. Rational numbers are commutative under addition.

       Example:

       $\frac{2}{3}+\left(\frac{-3}{2}\right)=\left(\frac{-3}{2}\right)+\left(\frac{2}{3}\right)=\frac{-5}{6}$

    2. Rational numbers are not commutative under subtraction.

       Example :
       $\left(\frac{3}{4}\right)-\left(\frac{5}{2}\right)=\left(\frac{-7}{4}\right)$ and $\frac{5}{2}-\frac{3}{4}=\frac{7}{4}$
       $\therefore\left(\frac{3}{4}\right)-\left(\frac{5}{2}\right)\neq\left(\frac{5}{2}\right)-\left(\frac{3}{4}\right)$

    3. Rational numbers are commutative under multiplication.

       Example:
       $\left(\frac{3}{4}\right)\times\left(\frac{-2}{6}\right)=\left(\frac{-2}{6}\right)\times\left(\frac{3}{4}\right)=\frac{-1}{4}$

    4. Rational numbers are not commutative under division.

       $2\div 5\neq 5\div 2$

- **Associativity**
  - Whole numbers are associative under addition and multiplication. However, they are **not** associative under subtraction and division.
  - Integers are associative under addition and multiplication. However, they are **not** associative under subtraction and division.
  - Rational numbers:
    1. Rational numbers are associative under addition.

       Example:
       $\left(\frac{2}{3}+\frac{1}{3}\right)+1=\frac{2}{3}+\left(\frac{1}{3}+1\right)=2$

    2. Rational numbers are **not** associative under subtraction.

       Example:
       $\left(\frac{2}{3}-\frac{1}{3}\right)-1=\frac{-2}{3}$
       $\frac{2}{3}-\left(\frac{1}{3}-1\right)=\frac{4}{3}$
       $\therefore\left(\frac{2}{3}-\frac{1}{3}\right)-1\neq\frac{2}{3}-\left(\frac{1}{3}-1\right)$

3. Rational numbers are associative under multiplication.

Example:
$\left(\frac{2}{3}\times\frac{1}{3}\right)\times 1=\frac{2}{3}\times\left(\frac{1}{3}\times 1\right)=\frac{2}{9}$

4. Rational numbers are **not** associative under division.

Example:
$\left\{\frac{2}{7}\div\left(\frac{-1}{14}\right)\right\}\div\frac{3}{7}=\frac{-28}{3}$
$\frac{2}{7}\div\left\{\left(\frac{-1}{14}\right)\div\frac{3}{7}\right\}=\frac{-12}{7}$
$\therefore\frac{2}{7}\div\left\{\left(\frac{-1}{14}\right)\right\}\div\frac{3}{7}\neq\frac{2}{7}\div\left\{\left(\frac{-1}{14}\right)\div\frac{3}{7}\right\}$

- 0 is the additive identity of whole numbers, integers, and rational numbers.

  $\therefore 0 + a = a + 0 = a$, where a is a rational number

- 1 is the multiplicative identity of whole numbers, integers, and rational numbers.

  $a \times 1 = 1 \times a = a$

- Additive inverse of a number is the number, which when added to a number, gives 0. It is also called the negative of a number.

  a + (– a) = (– a) + = 0

  ∴Additive inverse of $\frac{2}{5}$ is $\left(\frac{-2}{5}\right)$

- Reciprocal or multiplicative inverse of a number is the number, which when multiplied by the number, gives 1. Therefore, the reciprocal of a is $\frac{1}{a}$. $\left[a\times\frac{1}{a}=1\right]$

  ∴ Reciprocal of $\frac{-2}{3}$ is $\frac{-3}{2}$

- If x is a rational number with terminating decimal expansion then it can be expressed in the $\frac{p}{q}$ form, where p and q are co-prime (the HCF of p and q is 1) and the prime factorisation of q is of the form 2^n5^m, where n and m are non-negative integers.
- Let $x=\frac{p}{q}$ be any rational number.
  i. If the prime factorization of q is of the form 2^m5^n, where m and n are non-negative integers, then x has a terminating decimal expansion.
  ii. If the prime factorisation of q is not of the form 2^m5^n, where m and n are non-negative integers, then x has a non-terminating and repetitive decimal expansion.

For example, $\frac{17}{1600}=\frac{17}{2^6\times 5^2}$ has the denominator in the form 2^n5^m, where n = 6and m = 2 are non-negative integers. So, it has a terminating decimal expansion.

$\frac{723}{392}=\frac{3\times 241}{2^3\times 7^2}$ has the denominator not in the form 2^n5^m, where n and m are non-negative integers. So, it has a non-terminating decimal expansion.

- **Conversion of decimals into equivalent rational numbers:**
  - Non-terminating repeating decimals can be easily converted into their equivalent rational numbers.

For example, $2.35\overline{961}$ can be converted in the form $\frac{p}{q}$, where p and q are integers and $q \neq 0$ as follows:

Let $x = 2.35\overline{961}$

$\Rightarrow x = 2.35961961...$ … (1)

On multiplying both sides of equation (1) with 100, we obtain:

$100x = 235.961961961...$ … (2)

On multiplying both sides of equation (2) with 1000, we obtain:

$100000x = 235961.961961961...$ … (3)

On subtracting equation (2) from equation (3), we obtain:

$99900x = 235726$

$\Rightarrow x = \frac{235726}{99900} = \frac{117863}{49950}$

Thus, $2.35\overline{961} = \frac{117863}{49950}$

- **Representation of real numbers of the form $\sqrt{n}$ on the number line, where *n* is any positive real number:**

We cannot represent $\sqrt{n}$ on number line directly, so we will use the geometrical method to represent $\sqrt{n}$ on the number line.

Example:

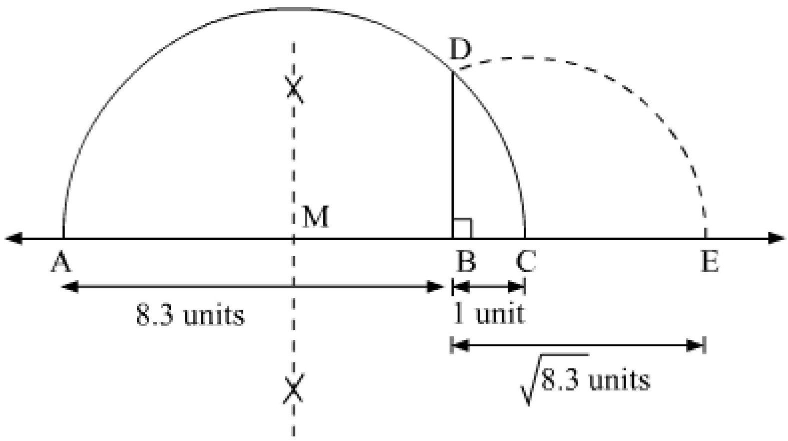
Represent $\sqrt{8.3}$ on the number line.

Solution:

Step 1: Draw a line and mark a point A on it. Mark points B and C such that AB = 8.3 units and BC = 1 unit.

Step 2: Find the mid-point of AC and mark it as M. Taking M as the centre and MA as the radius, draw a semi-circle.

Step 3: From B, draw a perpendicular to AC. Let it meet the semi-circle at D. Taking B as the centre and BD as the radius, draw an arc that intersects the line at E.



Now, the distance BE on this line is $\sqrt{8.3}$ units.

- Every number of the form $\sqrt{p}$, where p is a prime number is called an irrational number. For example, $\sqrt{3}$, $\sqrt{11}$, $\sqrt{12}$ etc.

Theorem: If a prime number p divides a^2, then p divides a, where a is a positive integer.

Example:
Prove that $\sqrt{7}$ is an irrational number.

Solution:
If possible, suppose $\sqrt{7}$ is a rational number.
Then, $\sqrt{7} = \frac{p}{q}$, where p, q are integers, $q \neq 0$.
If HCF $(p, q) \neq 1$, then by dividing p and q by HCF(p, q), $\sqrt{7}$ can be reduced as
$\sqrt{7} = \frac{a}{b}$ where HCF $(a, b) = 1$ … (1)
$\Rightarrow \sqrt{7}b = a$
$\Rightarrow 7b^2 = a^2$
$\Rightarrow a^2$ is divisible by 7
$\Rightarrow a$ is divisible by 7 … (2)
$\Rightarrow a = 7c$, where c is an integer
$\therefore \sqrt{7}c = b$
$\Rightarrow 7b^2 = 49c^2$
$\Rightarrow b^2 = 7c^2$
$\Rightarrow b^2$ is divisible by 7
$\Rightarrow b$ is divisible by 7 … (3)
From (2) and (3), 7 is a common factor of a and b. which contradicts (1)

$\therefore \sqrt{7}$ is an irrational number.

Example:
Show that $\sqrt{12} - 6$ is an irrational number.

Solution:
If possible, suppose $\sqrt{12} - 6$ is a rational number.
Then $\sqrt{12} - 6 = \frac{p}{q}$ for some integers p, q (q ¹ 0)
Now,

$$\sqrt{12} - 6 = \frac{p}{q}$$

$$\Rightarrow 2\sqrt{3} = \frac{p}{q} + 6$$

$$\Rightarrow \sqrt{3} = \frac{1}{2}\left(\frac{p}{q} + 6\right)$$

As p, q, 6 and 2 are integers, $\frac{1}{2}\left(\frac{p}{q} + 6\right)$ is rational number, so is $\sqrt{3}$.

This conclusion contradicts the fact that $\sqrt{3}$ is irrational.

Thus, $\sqrt{12} - 6$ is an irrational number.

- **Decimal expansion of irrational numbers**
    - The decimal expansion of an irrational number is non-terminating and non-repeating. Thus, a number whose decimal expansion is non-terminating and non- repeating is irrational.

For example, the decimal expansion of $\sqrt{2}$ is 1.41421…. , which is clearly non-terminating and non-repeating. Thus, $\sqrt{2}$ is an irrational number.

- The number $\sqrt[n]{a}$ is irrational if it is not possible to represent a in the form b^n, where b is a factor of a.

For example, $\sqrt[6]{12}$ is irrational as 12 cannot be written in the form b^6, where b is a factor of 12.

• If $\sqrt[a]{x}$ is an irrational number such that x is a positive rational number and a ($a \neq 1$) is a natural number, then $\sqrt[a]{x}$ is known as a **surd**. Here, $\sqrt{\ }$ is **radical** sign, a is **order** of the surd and x is **radicand**.

For example, $\sqrt[3]{10}$ is a surd of order 3.

• A surd whose order is 2 is called **quadratic surd**.

• **Rules for surds:**

If x, $y \in$ Q, x, $y > 0$ and a, b, $c \in$ N, then

(1) $\sqrt[a]{x} = (x)^{\frac{1}{a}}$

(2) $\left(\sqrt[a]{x}\right)^a = x$

(3) $\sqrt[a]{x}.\sqrt[a]{y} = \sqrt[a]{xy}$

(4) $\frac{\sqrt[a]{x}}{\sqrt[a]{y}} = \sqrt[a]{\frac{x}{y}}$

(5) $\sqrt[a]{\sqrt[b]{x}} = \sqrt[b]{\sqrt[a]{x}} = \sqrt[ab]{x}$

(6) $\sqrt[a]{x^b} = \sqrt[ac]{x^{bc}}$

(7) $\sqrt[a]{x^b} = \left(\sqrt[a]{x}\right)^b$

For example, $\sqrt[4]{\sqrt{8}} = \sqrt[4\times 2]{8} = \sqrt[8]{8}$ $\left[\sqrt[a]{\sqrt[b]{x}} = \sqrt[ab]{x}\right]$

Forms of surds

Pure form: A surd of the form $k\sqrt[a]{x}$ where $k \in$ Q such that $k = \pm 1$. For example, $\sqrt[3]{7}, -\sqrt{11}$ are pure surds.

Mixed form: A surd of the form $k\sqrt[a]{x}$ where $k \in Q$ such that $k \neq 0$ and $k \neq \pm 1$. For example, $3\sqrt[3]{5}, -4\sqrt{16}$ are mixed surds.

• **Conversion of mixed surds into pure surds:**

For example, $-5\sqrt{3} = -\sqrt{25}\sqrt{3} = -\sqrt{25 \times 3} = -\sqrt{75}$

• **Conversion of pure surds into mixed surds:**

For example, $\sqrt{27} = \sqrt{9 \times 3} = \sqrt{9} \times \sqrt{3} = \sqrt{3^2} \times \sqrt{3} = 3\sqrt{3}$

• In cases, where the radicand is a prime number or it has the factors whose roots are irrational, it is not possible to express pure surd as mixed surd.

For example, $\sqrt{11}, \sqrt{21}$ etc.

1. If $p\sqrt[a]{x}$ is a surd and q is another rational number, then $q\sqrt[a]{x}$ is called **similar surd** to $p\sqrt[a]{x}$ and vice versa.

2. If a surd is similar to another surd, then both the surds are called **like surds**.

For example, $5\sqrt[4]{15}, \frac{1}{2}\sqrt[4]{15}, -10\sqrt[4]{15}$ are like surds.

3. A surd of order a can be said into in its **simplest form** if it satisfies the following conditions:

(i) The radicand should not have any factor that can be represented as a^{th} power of a rational number.

(ii) The radicand should not be a fraction.

(iii) The surd should not be reducible to an order less than 'a' or 'a' should be the least order.

For example, $2\sqrt[3]{2}$ and $3\sqrt[3]{2}$ are in simplest form.

I. Having same order:

For the surds $\sqrt[a]{x}$ and $\sqrt[a]{y}$,

(i) if $x = y$, then $\sqrt[a]{x} = \sqrt[a]{y}$

(ii) if $x > y$, then $\sqrt[a]{x} > \sqrt[a]{y}$

(iii) if $x < y$, then $\sqrt[a]{x} < \sqrt[a]{y}$

II. Having different order:

To compare the surds $\sqrt[a]{x}$ and $\sqrt[b]{y}$ we need to convert both in such a way that both have same order.

i.e., $\sqrt[a]{x} = \sqrt[ab]{x^b}$ and $\sqrt[b]{y} = \sqrt[ab]{y^a}$

Now, compare the radicands to compare the surds.

• **Addition and subtraction of surds** can be done on like surds or by making them like surds.

For example, $\sqrt{12}+5\sqrt{12}=(1+5)\sqrt{12}=6\sqrt{12}$

• **Multiplication and division of surds** can be done by writing the surds as surds of same order.

For example, $\sqrt{6}\div\sqrt[3]{3}=\frac{\sqrt{6}}{\sqrt[3]{3}}=\frac{\sqrt[3\times2]{6^3}}{\sqrt[3\times2]{3^2}}=\frac{\sqrt[6]{216}}{\sqrt[6]{9}}=\sqrt[6]{\frac{216}{9}}=\sqrt[6]{24}$

• If a rational number is obtained after multiplying two surds then each surd is called the **rationalizing factor** of the other.

For example, $\sqrt{3}\times\sqrt{27}=\sqrt{3\times27}=\sqrt{81}=\sqrt{9^2}=9$

So, $\sqrt{3}$ and $\sqrt{27}$ are rationalizing factors of each other.

• The sum of a quadratic surd and a non-zero rational number or a quadratic surd is known as the binomial expression of quadratic surd.

For example, $8-3\sqrt{3}$ is a binomial expression of quadratic surds.

• The binomial expressions of quadratic surds of the form $\left(a+x\sqrt{b}\right)$ and $\left(a-x\sqrt{b}\right)$ are said to be **conjugate** of each other.

• $\left(a+x\sqrt{b}\right)$ is called the **rationalising factor** of $\left(a-x\sqrt{b}\right)$.

• **Some rules for equating:**

1. If $a+\sqrt{b}=x+\sqrt{y}$, where $a, b, x, y \in Q$ and $\sqrt{b}, \sqrt{y}$ are quadratic surds, then $a = x$ and $b = y$.

2. If $a+\sqrt{b}=\sqrt{y}$, where $a, b, y \in Q$ and $\sqrt{b}, \sqrt{y}$ are quadratic surds, then $a = 0$ and $b = y$.

• **Euclid's Division Lemma**

For any given positive integers a and b, there exists unique integers q and r such that $a = bq + r$ where $0 \le r < b$

Note: If b divides a, then $r = 0$

Example 1:

For $a = 15$, $b = 3$, it can be observed that
$15 = 3 \times 5 + 0$
Here, $q = 5$ and $r = 0$
If b divides a, then $0 < r < b$

Example 2:

For $a = 20$, $b = 6$, it can be observed that $20 = 6 \times 3 + 2$
Here, $q = 6$, $r = 2$, $0 < 2 < 6$

• **Euclid's division algorithm**

Euclid's division algorithm is a series of well-defined steps based on "Euclid's division lemma", to give a procedure for calculating problems.

Steps for finding HCF of two positive integers a and b ($a > b$) by using Euclid's division algorithm:

Step 1: Applying Euclid's division lemma to a and b to find whole numbers q and r, such that $a = bq + r,\ 0 \leq r < b$

Step 2: If $r = 0$, then HCF $(a, b) = b$

If $r \neq 0$, then again apply division lemma to b and r

Step 3: Continue the same procedure till the remainder is 0. The divisor at this stage will be the HCF of a and b.

Note: HCF (a, b) = HCF (b, r)

Example:

Find the HCF of 48 and 88.

Solution:

Take $a = 88$, $b = 48$

Applying Euclid's division lemma, we get

$88 = 48 \times 1 + 40$ (Here, $0 \leq 40 < 48$)

$48 = 40 \times 1 + 8$ (Here, $0 \leq 8 < 40$)

$40 = 8 \times 5 + 0$ (Here, $r = 0$)

HCF (48, 88) = 8

- **Using Euclid's division lemma to prove mathematical relationships**

Result 1:

Every positive even integer is of the form $2q$, while every positive odd integer is of the form $2q + 1$, where q is some integer.

Proof:

Let a be any given positive integer.

Take $b = 2$

By applying Euclid's division lemma, we have

$a = 2q + r$ where $0 \leq r < 2$

As $0 \leq r < 2$, either $r = 0$ or $r = 1$

If $r = 0$, then $a = 2q$, which tells us that a is an even integer.

If $r = 1$, then $a = 2q + 1$

It is known that every positive integer is either even or odd.

Therefore, a positive odd integer is of the form $2q + 1$.

Result 2:

Any positive integer is of the form $3q$, $3q + 1$ or $3q + 2$, where q is an integer.

Proof:

Let a be any positive integer.

Take $b = 3$

Applying Euclid's division lemma, we have

$a = 3q + r$, where $0 \leq r < 3$ and q is an integer

Now, $0 \leq r < 3$ Þ $r = 0, 1$, or 2

$\therefore a = 3q + r$

$\Rightarrow a = 3q + 0,\ a = 3q + 1,\ a = 3q + 2$

Thus, $a = 3q$ or $a = 3q + 1$ or $a = 3q + 2$, where q is an integer.

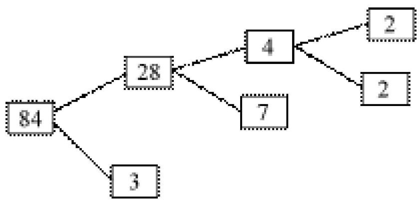
- Fundamental theorem of arithmetic states that very composite number can be uniquely expressed (factorised) as a product of primes apart from the order in which the prime factors occur.

Example: 1260 can be uniquely factorised as

2	1260
2	630
3	315
3	105
5	35
	7

$1260 = 2 \times 2 \times 3 \times 3 \times 5 \times 7$

Example: Factor tree of 84



$84 = 2 \times 2 \times 3 \times 7$

- For any positive integer a, b, **HCF (a, b) × LCM (a, b) = $a \times b$**

Example 1:
Find the LCM of 315 and 360 by the prime factorisation method. Hence, find their HCF.

Solution:
$315 = 3 \times 3 \times 5 \times 7 = 3^2 \times 5 \times 7$
$360 = 2 \times 2 \times 2 \times 3 \times 3 \times 5 = 2^3 \times 3^2 \times 5$
$\text{LCM} = 3^2 \times 5 \times 7 \times 2^3 = 2520$

$$\therefore \text{HCF}(315, 360) = \frac{315 \times 360}{\text{LCM}(315, 360)} = \frac{315 \times 360}{2520} = 45$$

Example 2:
Find the HCF of 300, 360 and 240 by the prime factorisation method.

Solution:
$300 = 2^2 \times 3 \times 5^2$
$360 = 2^3 \times 3^2 \times 5$
$240 = 2^4 \times 3 \times 5$
$\text{HCF}(300, 360, 240) = 2^2 \times 3 \times 5 = 60$